JOURNAL OF ALGEBRA **130**, 435–450 (1990)

Notes on Projectively Related Ideals and Residual Division

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Communicated by Richard G. Swan

Received September 19, 1988

Let $\mathbf{I} = (I_1, ..., I_g)$ and $\mathbf{J} = (J_1, ..., J_f)$ be finite collections of ideals of a Noetherian ring R. Then $\mathbf{I}$ and $\mathbf{J}$ are projectively related in case the integral closures of the ideals $I_1^{i_1} \cdots I_g^{i_g}$ and $J_1^{j_1} \cdots J_f^{j_f}$ are equal for some positive integers $i_1, ..., i_g$ and $j_1, ..., j_f$. Some basic properties of this relation are proved, and then these are applied (when either $g = 1$ or each I_i is regular) to give several necessary and sufficient conditions for the existence of an ideal K that is projectively related to $\mathbf{I}$ such that $K^n : G = K^n$ for all $n \geqslant 1$ and for all G in a given multiplicatively closed set Γ of nonzero ideals of R. It is then shown that if there exists such an ideal K in R, then for certain rings A related to R there exist ideals L projectively related to $\mathbf{I}A$ such that $L^n : GA = L^n$ for all $n \geqslant 1$ and for all $G \in \Gamma$. © 1990 Academic Press, Inc.

1. INTRODUCTION

Ideals I and J in a Noetherian ring R are projectively equivalent in case the integral closures of I^i and J^j are equal for some positive integers i and j. This is an equivalence relation on the set of ideals of R that was introduced by Samuel in [13], and it has proved to be very useful in many research problems in commutative algebra. In Section 2 we extend this definition to finite collections $\mathbf{I}$ and $\mathbf{J}$ of ideals of R. We show that this new relation is not an equivalence relation on the set of finite collections of ideals of R, but it does preserve asymptotic prime divisors, when the ideals have height at least one, and essential prime divisors, when the ideals are regular. (This new relation seems to be quite useful, and it has some other nice properties in common with projectively equivalence, but we do not pursue this in this paper).

As an application of the results of Section 2, we note that it is often

* Research on this paper was supported in part by the National Science Foundation, Grant DMS-8521058.

0021-8693/90 $3.00

important in commutative algebra to know when, for given ideals I and G in a ring, $I : G = I$. Our application of projectively related ideals, and the main result in this paper, is related to this. Specifically, for a given finite collection $\mathbf{I} = (I_1, ..., I_g)$ of ideals of a Noetherian ring R and for a given multiplicatively closed set Γ of nonzero ideals of R, if either $g = 1$ or each I_i is regular, then (3.3) gives several necessary and sufficient conditions for the existence of an ideal K that is a projective extension of $\mathbf{I}$ (see (2.1.5)) such that $K^n : G = K^n$ for all $n \geqslant 1$ and for all $G \in \Gamma$. Among these are: (a) no ideal in Γ is contained in any essential prime divisor of $\mathbf{I}$; (b) for each finite collection $\mathbf{K} = (K_1, ..., K_f)$ of ideals of R that is projectively related to $\mathbf{I}$ there exist positive integers $h_1, ..., h_f$ such that $K_1^{k_1+n_1} \cdots K_f^{k_f+n_f} : G = K_1^{n_1} \cdots K_f^{n_f}(K_1^{k_1} \cdots K_f^{k_f} : G)$ for all $k_i \geqslant h_i$ and $n_i \geqslant 0$ $(i = 1, ..., f)$ and for all $G \in \Gamma$; and, (c) condition (b) holds for some such finite collection $\mathbf{K}$ of ideals of R. It is then shown that these conditions are inherited by certain rings A related to R together with the extended collections of ideals $\mathbf{I}A$ and ΓA, namely: to localizations, to factor rings modulo prime divisors of zero, to finite integral extension rings, and, to faithfully flat Noetherian extension rings. Thus, if there exists such an ideal K in R, then there also exist ideals L in these other rings that are projectively related to $\mathbf{I}A$ such that $L^n : GA = L^n$ for all $n \geqslant 1$ and for all $G \in \Gamma$. One application of this shows that, although the images in R/z (where $z \in \mathrm{Ass}(R)$) of an R-sequence $b_1, ..., b_g$ need not be a prime sequence, for each $i = 0, 1, ..., g-1$ there exist ideals C_i which are projectively equivalent to $(b_1, ..., b_i) R/z$ and are such that $C_i^n : (b_{i+1} R/z) = C_i^n$ for all $n \geqslant 1$.

I am indebted to Steve McAdam for several helpful conversations concerning this paper.

2. Projectively Related Sets of Ideals

In this section we introduce projectively related sets of ideals and prove a few of their basic properties. To state and prove these results we need to specify some notational conventions, and we also need several definitions and a few known facts concerning them, so we begin with these.

If k is a positive integer, then $\mathbf{P}_k$ (resp., $\mathbf{N}_k$, $\mathbf{Z}_k$) is the set of all k-tuples of positive (resp., nonnegative, all) integers. If $\mathbf{n} = (n_1, ..., n_k) \in \mathbf{N}_k$, then $\mathbf{n}(i)$ denotes n_i, the ith component of $\mathbf{n}$, and it will be said that $\mathbf{n}$ is *large* in case each $\mathbf{n}(i)$ is large. If $\mathbf{m}$ and $\mathbf{n}$ are in $\mathbf{N}_k$ and h is a positive integer, then $\mathbf{m} + \mathbf{n}$, $\mathbf{m} - \mathbf{n}$, $\mathbf{mn}$, and $h\mathbf{n}$ are defined in the usual componentwise manner (but we will use $\mathbf{m} - \mathbf{n}$ only if $\mathbf{m} \geqslant \mathbf{n}$ (that is, $\mathbf{m}(i) \geqslant \mathbf{n}(i)$ for $i = 1, ..., k$)). Also, if $\mathbf{J} = (J_1, ..., J_k)$ (resp., $\mathbf{c} = (c_1, ..., c_k)$) is a collection of k ideals (resp., elements) in a ring related to R, then by $\mathbf{J}^{\mathbf{n}}$ (resp., $\mathbf{c}^{\mathbf{n}}$) we mean

$J_1^{\mathbf{n}(1)} \cdots J_k^{\mathbf{n}(k)}$ (resp., $c_1^{\mathbf{n}(1)} \cdots c_k^{\mathbf{n}(k)}$). Finally, $\mathbf{1}$ denotes $(1, ..., 1) \in \mathbf{P}_k$ (so, for example, $\mathbf{J}^{\mathbf{1}} = J_1 \cdots J_k$ and $\mathbf{c}^{\mathbf{n}-\mathbf{1}} = c_1^{\mathbf{n}(1)-1} \cdots c_k^{\mathbf{n}(k)-1}$, if $\mathbf{n} \in \mathbf{P}_k$).

(2.1). DEFINITION. Let R be a Noetherian ring, let I be an ideal of R, and let $\mathbf{I} = (I_1, ..., I_g)$ be a finite collection of ideals of R. Then:

(2.1.1). The *Rees ring* $\mathbf{R} = \mathbf{R}(R, I_1, ..., I_g)$ *of* R *with respect to* $I_1, ..., I_g$ is the $\mathbf{Z}_g$-graded subring $\mathbf{R} = R[u_1, ..., u_g, t_1 I_1, ..., t_g I_g]$ of $R[u_1, ..., u_g, t_1, ..., t_g]$, where $t_1, ..., t_g$ are indeterminates and $u_i = 1/t_i$ for $i = 1, ..., g$.

(2.1.2). The *integral closure* I_a *of* I *in* R is the ideal $I_a = \{x \in R;\ x$ satisfies an equation of the form $x^n + b_1 x^{n-1} + \cdots + b_n = 0$, where $b_i \in I^i$ for $i = 1, ..., g\}$.

(2.1.3). $\hat{A}^*(I) = \{p \cap R;\ p$ is a prime divisor of $(u^n\mathbf{R})_a$ for some $n \geqslant 1\}$ and $\mathbf{E}(I) = \{p \cap R;\ p$ is a prime divisor of $u\mathbf{R}$ and the completion of $\mathbf{R}_p$ contains a depth one prime divisor of zero$\}$, where $\mathbf{R} = \mathbf{R}(R, I)$ (see (2.1.1) and (2.1.2)). The members of $\hat{A}^*(I)$ (resp., $\mathbf{E}(I)$) are called the *asymptotic* (resp., *essential*) *prime divisors of* I.

(2.1.4). $\hat{A}^*(\mathbf{I}) = \{p \cap R;\ p$ is a prime divisor of $(\mathbf{u}^{\mathbf{n}}\mathbf{R})_a$ for some nonzero $\mathbf{n} \in \mathbf{N}_g\}$ and $\mathbf{E}(\mathbf{I}) = \{p \cap R;\ p$ is a prime divisor of $\mathbf{u}^{\mathbf{1}}\mathbf{R}$ and the completion of $\mathbf{R}_p$ contains a depth one prime divisor of zero$\}$, where $\mathbf{u} = (u_1, ..., u_g)$ and $\mathbf{R} = \mathbf{R}(R, I_1, ..., I_g)$ (see (2.1.1) and (2.1.2)). The members of $\hat{A}^*(\mathbf{I})$ (resp., $\mathbf{E}(\mathbf{I})$) are called the *asymptotic* (resp., *essential*) *prime divisors of* $\mathbf{I}$.

(2.1.5). If I and J are ideals in R, then I and J are *projectively equivalent* in case $(I^i)_a = (J^j)_a$ for some positive integers i and j. And, if $\mathbf{J} = (J_1, ..., J_f)$ is another finite collection of ideals of R, then $\mathbf{I}$ and $\mathbf{J}$ are said to be *projectively related* in case $(\mathbf{I}^{\mathbf{i}})_a = (\mathbf{J}^{\mathbf{j}})_a$ for some $\mathbf{i} \in \mathbf{P}_g$ and for some $\mathbf{j} \in \mathbf{P}_f$. Also, $\mathbf{J}$ is a *projective extension* of $\mathbf{I}$ in case $\mathbf{I}^{\mathbf{i}} \subseteq \mathbf{J}^{\mathbf{1}} \subseteq (\mathbf{I}^{\mathbf{i}})_a$ for some $\mathbf{i} \in \mathbf{P}_g$. (If $f = 1$, then we will say J_1 is projectively related to $\mathbf{I}$ (resp., J_1 is a projective extension of $\mathbf{I}$).)

It should be noted that (2.1.3) and (2.1.4) agree if $g = 1$. And concerning (2.1.5), it is clear that every projective extension of $\mathbf{I}$ is projectively related to $\mathbf{I}$ and that if $g = f = 1$, then I_1 and J_1 are projectively related if and only if they are projectively equivalent.

Remark (2.2) lists a few known facts concerning (2.1.3) that will be needed below.

(2.2). *Remark.* Let I be an ideal in a Noetherian ring R and let $\mathbf{I} = (I_1, ..., I_g)$ be a finite collection of ideals of R. Then:

(2.2.1). It is shown in [9, (2.1.5), (2.4), and (2.7)] that $\hat{A}^*(I) = \{P \in \mathrm{Spec}(R);\ P \in \mathrm{Ass}(R/(I^n)_a)$ for all large $n\} = \{P \in \mathrm{Spec}(R);\ P \in \mathrm{Ass}(R/(I^n)_a)$ for some $n \geqslant 1\}$.

(2.2.2). It is shown in [4, (1.3)] that if $P \in \text{Ass}(R/I_a)$, then $P \in \text{Ass}(R/(IJ)_a)$ for all ideals J of R such that $height(J) \geqslant 1$.

(2.2.3). If J is projectively equivalent to I, then $\mathbf{E}(J) = \mathbf{E}(I)$ by [3, (2.5.6)], and a proof similar to that in [3] shows that $\hat{A}^*(J) = \hat{A}^*(I)$.

(2.2.4). It is shown in [3, (2.5) and (2.11)] that the following hold: (a) If S is a multiplicatively closed set in R such that $0 \notin S$, then $\mathbf{E}(IR_S) = \{PR_S;\ P \in \mathbf{E}(I) \text{ and } P \cap S = \varnothing\}$. (b) $P \in \mathbf{E}(I)$ if and only if there exists $z \in \text{Ass}(R)$ such that $z \subseteq P$ and $P/z \in \mathbf{E}((I+z)/z)$. (c) If A is a Noetherian ring which is a faithfully flat R-algebra, then $\mathbf{E}(I) = \{P \cap R;\ P \in \mathbf{E}(IA)\}$, and if $P \in \mathbf{E}(I)$ and P^* is a minimal prime divisor of PA, then $P^* \in \mathbf{E}(IA)$. (d) If B is a finite integral extension ring of R, then $\mathbf{E}(I) \subseteq \{P' \cap R;\ P' \in \mathbf{E}(IB)\}$, and if $z \cap R \in \text{Ass}(R)$ for all $z \in \text{Ass}(B)$, then the equality holds. (e) $\hat{A}^*(I) \subseteq \mathbf{E}(I)$, and the equality holds if the completion of R_M has no embedded prime divisors of zero for all maximal ideals M in R that contain I.

Our first new result in this section extends (2.2.4) from I to $\mathbf{I}$.

(2.3). Proposition. *Let $\mathbf{I} = (I_1, ..., I_g)$ be a finite collection of ideals of a Noetherian ring R. Then*:

(2.3.1). *If S is a multiplicatively closed set in R such that $0 \notin S$, then $\mathbf{E}(I_1 R_S, ..., I_g R_S) = \{PR_S;\ P \in \mathbf{E}(\mathbf{I}) \text{ and } P \cap S = \varnothing\}$.*

(2.3.2). *$P \in \mathbf{E}(\mathbf{I})$ if and only if there exists $z \in \text{Ass}(R)$ such that $z \subseteq P$ and $P/z \in \mathbf{E}((I_1+z)/z, ..., (I_g+z)/z)$.*

(2.3.3). *If A is a Noetherian ring which is a faithfully flat R-algebra, then $\mathbf{E}(\mathbf{I}) = \{P \cap R;\ P \in \mathbf{E}(I_1 A, ..., I_g A)\}$, and if $P \in \mathbf{E}(\mathbf{I})$ and P^* is a minimal prime divisor of PA, then $P^* \in \mathbf{E}(I_1 A, ..., I_g A)$.*

(2.3.4). *If B is a finite integral extension ring of R, then $\mathbf{E}(\mathbf{I}) \subseteq \{P' \cap R;\ P' \in \mathbf{E}(I_1 B, ..., I_g B)\}$, and if $z \cap R \in Ass(R)$ for all $z \in Ass(B)$, then the equality holds.*

(2.3.5). *$\hat{A}^*(\mathbf{I}) \subseteq \mathbf{E}(\mathbf{I})$, and the equality holds if the completion of R_M has no embedded prime divisors of zero for all maximal ideals M in R that contain I_i for some $i = 1, ..., g$.*

Proof. The proofs in [3] of (2.2.4)(a)–(e) use the ring $\mathbf{R}(R, I)$ and some basic properties of the quintessential prime divisors of an ideal J applied to $u\mathbf{R}(R, I)$. (The quintessential prime divisors of J are the ideals in $\mathbf{Q}(J) = \{P \in \text{Spec}(R);\ J \subseteq P$ and $J(R_P)^* + z$ is $P(R_P)^*$-primary for some prime divisor z of zero in the completion $(R_P)^*$ of $R_P\}$.) By using $\mathbf{R} = \mathbf{R}(R, I_1, ..., I_g)$ and $\mathbf{u}^1\mathbf{R}$ in place of $\mathbf{R}(R, I)$ and $u\mathbf{R}(R, I)$, respectively, essentially the same proof show that (2.3.1)–(2.3.5) hold. Q.E.D.

It is well known that "I is projectively equivalent to J" is an equivalence relation on the set of ideals of R. In (2.4) we show that "$\mathbf{I}$ is projectively related to $\mathbf{J}$" is reflexive and symmetric, but it is not transitive on the set of finite collections of ideals of R.

(2.4). PROPOSITION. *Let $\mathbf{I} = (I_1, ..., I_g)$ be a finite collection of ideals of a Noetherian ring R. Then $\mathbf{I}^{[\mathbf{m}]}$ is projectively related to $\mathbf{I}^{[\mathbf{n}]}$ for all $\mathbf{m}$ and $\mathbf{n}$ in $\mathbf{P}_g$, so, in particular, projectively related is reflexive. And it is also symmetric, but it is not transitive. (Here, $\mathbf{I}^{[\mathbf{m}]} = (I_1^{\mathbf{m}(1)}, ..., I_g^{\mathbf{m}(g)})$.)*

Proof. It is clear that projectively related is reflexive and symmetric, and $(\mathbf{I}^{[\mathbf{m}]})^{\mathbf{n}} = (\mathbf{I}^{[\mathbf{n}]})^{\mathbf{m}}$ for all $\mathbf{m}$ and $\mathbf{n}$ in $\mathbf{N}_g$, so $((\mathbf{I}^{[\mathbf{m}]})^{\mathbf{n}})_a = ((\mathbf{I}^{[\mathbf{n}]})^{\mathbf{m}})_a$, hence $\mathbf{I}^{[\mathbf{m}]}$ and $\mathbf{I}^{[\mathbf{n}]}$ are projectively related.

To complete the proof we will give an example of three collections of ideals which do not satisfy the transitive law. For this let (R, M) be a regular local ring such that *altitude*$(R) \geqslant 2$ and let b be a nonzero nonunit in R. Also, let $\mathbf{I} = (bM)$, $\mathbf{J} = (bR, M)$, and $\mathbf{K} = (b^2M)$. Then $(\mathbf{I}^1)_a = (bM)_a = (\mathbf{J}^{(1,1)})_a$ and $(\mathbf{J}^{(2,1)})_a = (b^2M)_a = (\mathbf{K}^1)_a$. Therefore $\mathbf{I}$ is projectively related to $\mathbf{J}$ and $\mathbf{J}$ is projectively related to $\mathbf{K}$, so it remains to show that $(\mathbf{I}^m)_a \neq (\mathbf{K}^n)_a$ for all positive integers m and n; that is, that bM and b^2M are not projectively equivalent, and this is readily checked. Q.E.D.

In view of (2.4) it might be though that projectively related ideals will not be very useful. However, this is not the case, and to (partly) show this we show in the remainder of this section that if $\mathbf{I}$ and $\mathbf{J}$ are projectively related, then $\hat{A}^*(\mathbf{I}) = \hat{A}^*(\mathbf{J})$ (if the ideals have height at least one) and $\mathbf{E}(\mathbf{I}) = \mathbf{E}(\mathbf{J})$ (if the ideals are regular).

(2.5). THEOREM. *If $\mathbf{I} = (I_1, ..., I_g)$ and $\mathbf{J} = (J_1, ..., J_f)$ are projectively related collections of ideals of a Noetherian ring R such that height$(I_i) \geqslant 1$ for $i = 1, ..., g$, then:*

(2.5.1). *$\hat{A}^*(\mathbf{I}) = \hat{A}^*(I_1 \cdots I_g) = \bigcup \{\hat{A}^*(\mathbf{I}^{\mathbf{m}});\ \mathbf{m}$ is nonzero in $\mathbf{N}_g\} = \hat{A}^*(\mathbf{I}^{\mathbf{n}})$ for all $\mathbf{n} \in \mathbf{P}_g$.*

(2.5.2). *$\hat{A}^*(\mathbf{I}) = \hat{A}^*(\mathbf{J})$.*

Proof. For (2.5.1) it is clear that $\hat{A}^*(\mathbf{I}^{\mathbf{n}}) \subseteq \bigcup \{\hat{A}^*(\mathbf{I}^{\mathbf{m}});\ \mathbf{m}$ is nonzero in $\mathbf{N}_g\}$ for all $\mathbf{n} \in \mathbf{P}_g$. Also, if $P \in \hat{A}^*(I_1 \cdots I_g)$, then $P \in \mathrm{Ass}(R/((\mathbf{I}^1)^k)_a)$ for all large k by (2.2.1), so since *height*$(I_i) \geqslant 1$ for $i = 1, ..., g$ it follows from (2.2.2) that $P \in \mathrm{Ass}(R/((\mathbf{I}^{[\mathbf{n}]})^k)_a)$ for all $\mathbf{n} \in \mathbf{P}_g$, so $P \in \hat{A}^*(\mathbf{I}^{\mathbf{n}})$ by (2.2.1), hence $\hat{A}^*(\mathbf{I}^1) \subseteq \hat{A}^*(\mathbf{I}^{\mathbf{n}})$ for all $\mathbf{n} \in \mathbf{P}_g$. And if $P \in \hat{A}^*(\mathbf{I}^{\mathbf{m}})$ for some nonzero $\mathbf{m} \in \mathbf{N}_g$, then $P \in \mathrm{Ass}(R/((\mathbf{I}^{\mathbf{m}})^n)_a)$ for all large n by (2.2.1), so it follows from (2.2.2) that if $k \geqslant \max\{n\mathbf{m}(i);\ i = 1, ..., g\}$, then $P \in \mathrm{Ass}(R/(\mathbf{I}^{\mathbf{k}})_a)$, where $\mathbf{k}(i) = k$ for $i = 1, ..., g$. But $\mathbf{I}^{\mathbf{k}} = (\mathbf{I}^1)^k$, so it follows from (2.2.1) that

$P \in \hat{A}^*(I_1 \cdots I_g)$, hence $\bigcup \{\hat{A}^*(\mathbf{I}^\mathbf{m})$; $\mathbf{m}$ is nonzero in $\mathbf{N}_g\} \subseteq \hat{A}^*(I_1 \cdots I_g)$, so $\hat{A}^*(I_1 \cdots I_g) = \bigcup \{\hat{A}^*(\mathbf{I}^\mathbf{m})$; $\mathbf{m}$ is nonzero in $\mathbf{N}_g\} = \hat{A}^*(\mathbf{I}^\mathbf{n})$ for all $\mathbf{n} \in \mathbf{P}_g$.

To complete the proof of (2.5.1) it will next be shown that:

(2.5.3). If $\mathbf{H} = (H_1, ..., H_h)$ is a collection of $h \geqslant 1$ ideals of a Noetherian ring R such that $height(H_i) \geqslant 1$ for $i = 1, ..., h$, and if $p \in \mathrm{Ass}(\mathbf{B}/(\mathbf{u}^\mathbf{k}\mathbf{B})_a)$ for some $\mathbf{k} \in \mathbf{P}_h$, where $\mathbf{B} = \mathbf{R}(R, H_1, ..., H_h)$ and $\mathbf{u} = (u_1, ..., u_h)$, then $p \cap R \in \mathrm{Ass}(R/(\mathbf{H}^\mathbf{m})_a)$ for all large $\mathbf{m} \in \mathbf{P}_h$.

For this, if $h = 1$, then the conclusion holds by (2.2.1) and (2.1.3). Therefore assume that $h > 1$ and that the conclusion holds for collections of $h - 1$ ideals. Let $p \in \mathrm{Ass}(\mathbf{B}/(\mathbf{u}^\mathbf{k}\mathbf{B})_a)$ and consider the two cases: (a) $u_i \notin p$ for some $i = 1, ..., h$; and, (b) $u_i \in p$ for $i = 1, ..., h$.

If (a) holds, then let $\mathbf{A} = R[u_1, ..., u_{i-1}, u_{i+1}, ..., u_h, t_1 H_1, ..., t_{i-1} H_{i-1}, t_{i+1} H_{i+1}, ..., t_h H_h]$. Then $\mathbf{A}[u_i, t_i] = \mathbf{B}[1/u_i]$ and $p\mathbf{B}[1/u_i] \in \mathrm{Ass}(\mathbf{B}[1/u_i]/(\mathbf{u}^\mathbf{k}\mathbf{B}[1/u_i])_a)$, so since t_i is an indeterminate and $u_i = 1/t_i$ it follows that $p\mathbf{B}[1/u_i] = q\mathbf{A}[u_i, t_i]$ for some $q \in \mathrm{Ass}(\mathbf{A}/(\mathbf{v}^\mathbf{j}\mathbf{A})_a)$, where $\mathbf{v} = (u_1, ..., u_{i-1}, u_{i+1}, ..., u_k)$ and $\mathbf{j} \in \mathbf{P}_{h-1}$ is such that $\mathbf{j}(l) = \mathbf{k}(l)$ for $l = 1, ..., i-1$ and $\mathbf{j}(l) = \mathbf{k}(l+1)$ for $l = i, ..., h-1$, and it then follows that $q = p\mathbf{B}[1/u_i] \cap \mathbf{A}$. Therefore by induction on $h \geqslant 1$ it follows that $p \cap R = q \cap R \in \mathrm{Ass}(R/(\mathbf{G}^\mathbf{m})_a)$ for all large $\mathbf{m} \in \mathbf{P}_{h-1}$, where $\mathbf{G} = (H_1, ..., H_{i-1}, H_{i+1}, ..., H_h)$, so it follows form (2.2.2) that $p \cap R \in \mathrm{Ass}(R/(\mathbf{H}^\mathbf{m})_a)$ for all large $\mathbf{m} \in \mathbf{P}_h$.

If (b) holds, then it is shown in [1, Proposition 1, p. 283] that there exists a homogeneous element $bt^\mathbf{n} \in \mathbf{B}$ such that $p = (\mathbf{u}^\mathbf{k}\mathbf{B})_a : bt^\mathbf{n}\mathbf{B}$. Now possibly $\mathbf{n}(i)$ is negative for some $i = 1, ..., h$. If this is the case, then let $\mathbf{A}$ be as in the preceding paragraph, so $\mathbf{B} = \mathbf{A}[u_i, t_i H_i] \cong \mathbf{R}(\mathbf{A}, H_i\mathbf{A})$. Also, since each u_i is regular in $\mathbf{B}$ it follows that $p \in \mathrm{Ass}(\mathbf{B}/(u_i^k\mathbf{B})_a)$ for some $k \geqslant 1$, so it is shown in [8, (3.2)(1)] (from the third sentence on) that $t_i H_i \nsubseteq p$. Thus there exists an element $c \in H_i$ such that $ct_i \notin p$, so $p = (\mathbf{u}^\mathbf{k}\mathbf{B})_a : bt^\mathbf{n}(ct_i)^n\mathbf{B}$ for all $n \geqslant 1$. By repeating this for each negative coordinate of $\mathbf{n}$ it may be assumed that $p = (\mathbf{u}^\mathbf{k}\mathbf{B})_a : dt^\mathbf{m}\mathbf{B}$ for some nonzero $\mathbf{m} \in \mathbf{N}_h$ and for some $d \in \mathbf{H}^\mathbf{m}$. Now a brief computation shows that $(\mathbf{u}^\mathbf{k}\mathbf{B})_a : dt^\mathbf{m}\mathbf{B} \cap R = (\mathbf{H}^{\mathbf{m}+\mathbf{k}})_a : dR$. (Specifically, let $\mathbf{B}'$ be the integral closure of $\mathbf{B}$. Then $r \in (\mathbf{H}^{\mathbf{m}+\mathbf{k}})_a : dR$ if and only if $rd \in (\mathbf{H}^{\mathbf{m}+\mathbf{k}})_a = \mathbf{u}^{\mathbf{m}+\mathbf{k}}\mathbf{B}' \cap R$ if and only if (since $dt^\mathbf{m} \in \mathbf{B}$) $rdt^\mathbf{m} \in \mathbf{u}^\mathbf{k}\mathbf{B}' \cap \mathbf{B}$ if and only if $r \in (\mathbf{u}^\mathbf{k}\mathbf{B})_a : dt^\mathbf{m}\mathbf{B} \cap R$.) Therefore if follows that $p \cap R = (\mathbf{H}^{\mathbf{m}+\mathbf{k}})_a : dR$, so it follows from (2.2.2) that $p \cap R \in \mathrm{Ass}(R/(\mathbf{H}^\mathbf{m})_a)$ for all large $\mathbf{m} \in \mathbf{P}_h$.

Now to complete the proof of (2.5.1), it follows from (2.5.3) that if $P \in \hat{A}^*(\mathbf{I})$, say $P = p \cap R$ with $p \in \mathrm{Ass}(\mathbf{R}/(\mathbf{u}^\mathbf{m}\mathbf{R})_a)$ (where $\mathbf{R} = \mathbf{R}(R, I_1, ..., I_g)$ and $\mathbf{m} \in \mathbf{N}_g$ (see (2.1.4))), then $P \in \mathrm{Ass}(R/(\mathbf{I}^\mathbf{n})_a)$ for all large $\mathbf{n} \in \mathbf{P}_g$, so $P \in \hat{A}^*(\mathbf{I}^\mathbf{n})$ by (2.2.1). Therefore $\hat{A}^*(\mathbf{I}) \subseteq \hat{A}^*(\mathbf{I}^\mathbf{n})$. And if $P \in \hat{A}^*(\mathbf{I}^\mathbf{m})$ for some nonzero $\mathbf{m} \in \mathbf{N}_g$, then $P \in \mathrm{Ass}(R/((\mathbf{I}^\mathbf{m})^k)_a)$ for all large k by (2.2.1). But $(\mathbf{I}^{k\mathbf{m}})_a = (\mathbf{u}^{k\mathbf{m}}\mathbf{R})_a \cap R$, so there exists a prime divisor p of $(\mathbf{u}^{k\mathbf{m}}\mathbf{R})_a$ such that

$p \cap R = P$, hence $P \in \hat{A}^*(\mathbf{I})$ by (2.1.4). Therefore it follows that $\hat{A}^*(\mathbf{I}) = \bigcup \{\hat{A}^*(\mathbf{I}^{\mathbf{m}});\ \mathbf{m}$ is nonzero in $\mathbf{N}_g\}$, so (2.5.1) holds.

For (2.5.2), by hypothesis $(\mathbf{I}^{\mathbf{i}})_a = (\mathbf{J}^{\mathbf{j}})_a$ for some $\mathbf{i} \in \mathbf{P}_g$ and for some $\mathbf{j} \in \mathbf{P}_f$, so it follows that $\mathrm{Rad}(I_1 \cdots I_g) = \mathrm{Rad}(J_1 \cdots J_f)$. Therefore the hypothesis $height(I_i) \geqslant 1$ for $i = 1, ..., g$ implies that $height(J_j) \geqslant 1$ for $j = 1, ..., f$, so (2.5.1) applied to $\mathbf{J}$ shows that $\hat{A}^*(\mathbf{J}) = \hat{A}^*(\mathbf{J}^{\mathbf{j}})$ for all $\mathbf{j} \in \mathbf{P}_f$. Also, $\hat{A}^*(K) = \hat{A}^*(K_a)$ holds for all ideals K, by (2.2.3), so it follows from this and (2.5.1) (applied to both $\mathbf{I}$ and $\mathbf{J}$) that $\hat{A}^*(\mathbf{I}) = \hat{A}^*(\mathbf{I}^{\mathbf{i}}) = \hat{A}^*((\mathbf{I}^{\mathbf{i}})_a) = \hat{A}^*((\mathbf{J}^{\mathbf{j}})_a) = \hat{A}^*(\mathbf{J}^{\mathbf{j}}) = \hat{A}^*(\mathbf{J})$. Q.E.D.

Theorem (2.6) is the essential prime divisor analog of (2.5).

(2.6). THEOREM. *If* $\mathbf{I} = (I_1, ..., I_g)$ *and* $\mathbf{J} = (J_1, ..., J_h)$ *are projectively related collections of ideals of a Noetherian ring* R *such that each* I_i *is regular, then*:

(2.6.1). $\mathbf{E}(\mathbf{I}) = \mathbf{E}(I_1 \cdots I_g) = \bigcup \{\mathbf{E}(\mathbf{I}^{\mathbf{m}});\ \mathbf{m}$ *is nonzero in* $\mathbf{N}_g\} = \mathbf{E}(\mathbf{I}^{\mathbf{n}})$ *for all* $\mathbf{n} \in \mathbf{P}_g$.

(2.6.2). $\mathbf{E}(\mathbf{I}) = \mathbf{E}(\mathbf{J})$.

Proof. For (2.6.1) assume first that R is locally unmixed, so $\mathbf{E}(I) = \hat{A}^*(I)$ for all ideals I in R by (2.2.4)(e). Therefore $\mathbf{E}(\mathbf{I}^{\mathbf{m}}) = \hat{A}^*(\mathbf{I}^{\mathbf{m}})$ for all nonzero $\mathbf{m} \in \mathbf{N}_g$, so it follows from (2.5.1) that $\mathbf{E}(\mathbf{I}^1) = \bigcup \{\mathbf{E}(\mathbf{I}^{\mathbf{m}});\ \mathbf{m}$ is nonzero in $\mathbf{N}_g\} = \mathbf{E}(\mathbf{I}^{\mathbf{n}})$ for all $\mathbf{n} \in \mathbf{P}_g$. Also, it follows from [7, Corollary, p. 61] that $\mathbf{R} = \mathbf{R}(R, I_1, ..., I_g)$ is locally unmiwed, so it readily follows that $\hat{A}^*(\mathbf{u}^1\mathbf{R}) = \{p;\ p$ is a height one prime divisor of $\mathbf{u}^1\mathbf{R}\} = \{p \in \mathrm{Ass}(\mathbf{R}/u^1\mathbf{R});$ there exists a depth one prime divisor of zero in the completion of $\mathbf{R}_p\}$. It therefore follows from (2.1.4) that $\mathbf{E}(\mathbf{I}) = \hat{A}^*(\mathbf{I})$, so it follows from (2.5.1) that $\mathbf{E}(\mathbf{I}) = \hat{A}^*(\mathbf{I}) = \hat{A}^*(\mathbf{I}^1)$, and it has already been noted that $\hat{A}^*(\mathbf{I}^1) = \mathbf{E}(\mathbf{I}^1)$, so (2.6.1) holds when R is locally unmixed.

For the general case let $P \in \mathbf{E}(\mathbf{I})$, so $PR_P \in \mathbf{E}(\mathbf{I}R_P)$ by (2.3.1). Also, if L is the completion of R_P, then $PL \in \mathbf{E}(\mathbf{I}L)$ by (2.3.3), so (2.3.2) shows that there exists $z \in \mathrm{Ass}(L)$ such that $z \subseteq PL$ and $PL/z \in \mathbf{E}((\mathbf{I}L + z)/z)$. But L/z is a complete local domain, so it follows from the preceding paragraph that $PL/z \in \mathbf{E}(\mathbf{I}^1 L + z)/z) = \bigcup \{\mathbf{E}((\mathbf{I}^{\mathbf{m}}L + z)/z);\ \mathbf{m}$ is nonzero in $\mathbf{N}_g\} = \mathbf{E}((\mathbf{I}^n L + z)/z)$ for all $\mathbf{n} \in \mathbf{P}_g$. Therefore (2.3.2) implies that $PL \in \mathbf{E}(\mathbf{I}^1 L) \cap (\bigcup \{\mathbf{I}^{\mathbf{m}}L;\ \mathbf{m}$ is nonzero in $\mathbf{N}_g\}) \cap \mathbf{E}(\mathbf{I}^{\mathbf{n}}L)$ for all $\mathbf{n} \in \mathbf{P}_g$. Then (2.3.3) shows that $PR_P \in \mathbf{E}(\mathbf{I}^1 R_P) \cap (\bigcup \{\mathbf{I}^{\mathbf{m}}R_P;\ \mathbf{m}$ is nonzero in $\mathbf{N}_g\}) \cap \mathbf{E}(\mathbf{I}^{\mathbf{n}}R_P)$ for all $\mathbf{n} \in \mathbf{P}_g$, and so (2.3.1) shows that $P \in \mathbf{E}(\mathbf{I}^1) \cap (\bigcup \{\mathbf{E}(\mathbf{I}^{\mathbf{m}});\ \mathbf{m}$ is nonzero in $\mathbf{N}_g\}) \cap \mathbf{E}(\mathbf{I}^{\mathbf{n}})$ for all $\mathbf{n} \in \mathbf{P}_g$. Thus $\mathbf{E}(\mathbf{I})$ is a subset of the other three sets in (2.6.1). And it is similarly seen that each of these three sets is a subset of $\mathbf{E}(\mathbf{I})$, so all four sets are equal.

For (2.6.2), by hypothesis $(\mathbf{I}^{\mathbf{i}})_a = (\mathbf{J}^{\mathbf{j}})_a$ for some $\mathbf{i} \in \mathbf{P}_g$ and for some $\mathbf{j} \in \mathbf{P}_f$, so $\mathrm{Rad}(I_1 \cdots I_g) = \mathrm{Rad}(J_1 \cdots J_f)$. Therefore the hypothesis that each

I_i is regular implies that each J_j is regular, so (2.6.1) applied to $\mathbf{J}$ shows that $\mathbf{E}(\mathbf{J}) = \mathbf{E}(\mathbf{J}^{\mathbf{j}})$ for all $\mathbf{j} \in \mathbf{P}_f$. Also, $\mathbf{E}(K) = \mathbf{E}(K_a)$ holds for all ideals K by (2.2.3), so it follows from this and (2.6.1) that $\mathbf{E}(\mathbf{I}) = \mathbf{E}(\mathbf{I}^{\mathbf{i}}) = \mathbf{E}((\mathbf{I}^{\mathbf{i}})_a) = \mathbf{E}((\mathbf{J}^{\mathbf{j}})_a) = \mathbf{E}(\mathbf{J}^{\mathbf{j}}) = \mathbf{E}(\mathbf{J})$. Q.E.D.

Corollary (2.7) is of some interest since the transitivity property does not hold for projectively related ideals.

(2.7). COROLLARY. *Let* $\mathbf{H} = (H_1, ..., H_h)$, $\mathbf{I} = (I_1, ..., I_g)$, *and* $\mathbf{J} = (J_1, ..., J_f)$ *be finite collections of ideals of a Noetherian ring R such that* $\mathbf{H}$ *is projectively related to* $\mathbf{I}$ *and* $\mathbf{I}$ *is projectively related to* $\mathbf{J}$. *Then*:

(2.7.1). *If each* I_i *has eight at least one, then* $\hat{A}^*(\mathbf{H}) = \hat{A}^*(\mathbf{J})$.

(2.7.2). *If each* I_i *is regular, then* $\mathbf{E}(\mathbf{H}) = \mathbf{E}(\mathbf{J})$.

Proof. (2.7.1) (resp., (2.7.2)) readily follows from (2.5.2) (resp., (2.6.2)). Q.E.D.

3. ESSENTIAL PRIME DIVISORS AND RESIDUAL DIVISION

In this section we apply (2.5) and (2.6) to the following problem: Given a finite collection $\mathbf{I} = (I_1, ..., I_g)$ of ideals of a Noetherian ring R and a multiplicatively closed set Γ of nonzero ideals of R, when does there exist an ideal K that is projectively related to $\mathbf{I}$ such that $K^n : G = K^n$ for all $n \geqslant 1$ and for all $G \in \Gamma$? For this, we need one more definition and two more results concerning $\mathbf{R}(R, I_1, ..., I_g)$, so we begin with these.

(3.1). DEFINITION. If Γ is a multiplicatively closed set of nonzero ideals of a ring R and I is an ideal of R, then $I : \Gamma$ denotes the ideal $\bigcup \{I : G; G \in \Gamma\}$.

It should be noted that $I : \Gamma$ is of some interest, since, for example, if Q is a primary ideal in a Noetherian ring and Γ is the set of all finite products of the prime ideals properly containing $\mathrm{Rad}(Q)$, then $Q^n : \Gamma$ is the nth symbolic power $Q^{(n)}$ of Q^n.

Also concerning (3.1), note that if G_1 and G_2 are in Γ, then $I : G_1 \cup I : G_2 \subseteq I : G_1G_2$ and $G_1G_2 \in \Gamma$. Therefore it follows that $I : \Gamma$ is an ideal in R, so since R is Noetherian it follows that $I : \Gamma = I : G$ for some $G \in \Gamma$. Further, it is clear that $(I : \Gamma) : \Gamma = I : \Gamma$. And, finally, if $\bigcap \{Q_i; i = 1, ..., n\}$ is a normal primary decomposition of I and if the n ideals $\mathrm{Rad}(Q_i)$ are ordered so that $\{i$; there exists an ideal $G \in \Gamma$ such that $G \subseteq \mathrm{Rad}(Q_i)$ for some $i = 1, ..., n\} = \{m + 1, ..., n\}$ (possibly this is the empty set, or it may be $\{1, ..., n\}$), then it is readily seen that $I : \Gamma = \bigcap \{Q_i; i = 1, ..., m\}$ (so possibly $I : \Gamma = I$ or $I : \Gamma = R$).

(3.2). *Remark.* Let $\mathbf{I} = (I_1, ..., I_g)$ be a collection of g ideals of a Noetherian ring R. Then:

(3.2.1). If Γ is a multiplicatively closed set of nonzero ideals of R, then it is readily checked that $(\mathbf{I}^{\mathbf{m}} : \Gamma)(\mathbf{I}^{\mathbf{n}} : \Gamma) \subseteq \mathbf{I}^{\mathbf{m}+\mathbf{n}} : \Gamma$ for all $\mathbf{m}$, $\mathbf{n} \in \mathbf{N}_g$. Therefore it follows that if $t_1, ..., t_g$ are indeterminates and $u_i = 1/t_i$ (for $i = 1, ..., g$), then $\mathbf{S} = R[u_1, ..., u_g, \{\mathbf{t}^{\mathbf{n}}(\mathbf{I}^{\mathbf{n}} : \Gamma)\}_{\mathbf{n} \in \mathbf{N}_g}]$ is a $\mathbf{Z}_g$-graded subring of $R[u_1, ..., u_g, t_1, ..., t_g]$ whose homogeneous elements of degree $\mathbf{n} \in \mathbf{N}_g$ are the elements of the form $x\mathbf{t}^{\mathbf{n}}$ with $x \in \mathbf{I}^{\mathbf{n}} : \Gamma$. (Here, $\mathbf{t} = (t_1, ..., t_g)$; see the comment preceding (2.1).)

(3.2.2). Let $\mathbf{R} = \mathbf{R}(R, I_1, ..., I_g)$, let $\mathbf{P}$ be a set of associated primes of $\mathbf{u}^1\mathbf{R}$ (where $\mathbf{u} = (u_1, ..., u_g)$), let S be the set of regular elements in $\mathbf{R} - \bigcup \{p; p \in \mathbf{P}\}$, and let $\mathbf{T} = \mathbf{R}_S \cap \mathbf{R}[1/\mathbf{u}^1]$. Then $\mathbf{T}$ is a $\mathbf{Z}_g$-graded subring of $R[u_1, ..., u_g, t_1, ..., t_g]$ and $\mathbf{T} \supseteq R[u_1, ..., u_g, \{\mathbf{t}^{\mathbf{n}}\mathbf{I}_{\mathbf{n}}\}_{\mathbf{n} \in \mathbf{N}_g}]$, where $\mathbf{t} = (t_1, ..., t_g)$ and $\mathbf{I}_{\mathbf{n}} = \mathbf{u}^{\mathbf{n}}\mathbf{T} \cap R$. Moreover, if $\mathbf{P} = \{p \in \mathrm{Ass}(\mathbf{R}/\mathbf{u}^1\mathbf{R})$; there exists a depth one prime divisor of zero in the completion of $\mathbf{R}_p\}$, then $\mathbf{T}$ is a finite module over $\mathbf{R}$.

Proof of (3.2.2). Note first that $\mathbf{R}[1/\mathbf{u}^1] = R[u_1, ..., u_g, t_1, ..., t_g]$. Therefore if $x \in \mathbf{T}$, then there exist finitely many elements $\mathbf{i} \in \mathbf{Z}_g$ and $r_{\mathbf{i}} \in R$ such that $x = \sum r_{\mathbf{i}}\mathbf{t}^{\mathbf{i}}$. Also, $x \in \mathbf{R}[1/\mathbf{u}^1]$, so there exist $\mathbf{n} \in \mathbf{N}_g$ and $f \in \mathbf{R}$ such that $x = f/\mathbf{u}^{\mathbf{n}}$, and $x \in \mathbf{R}_S$, hence $f = \sum r_{\mathbf{i}}\mathbf{t}^{\mathbf{i}-\mathbf{n}} = \mathbf{u}^{\mathbf{n}}x \in \mathbf{u}^{\mathbf{n}}\mathbf{R}_S \cap \mathbf{R}$. Now $\mathbf{R}$ is a $\mathbf{Z}_g$-graded subring of $R[u_1, ..., u_g, t_1, ..., t_g]$ and $\mathbf{u}^{\mathbf{n}}\mathbf{R}$ is homogeneous, so $\mathbf{u}^{\mathbf{n}}\mathbf{R}_S \cap \mathbf{R}$ is the intersection of the (homogeneous) primary components of $\mathbf{u}^{\mathbf{n}}\mathbf{R}$ that are disjoint from S, so $\mathbf{u}^{\mathbf{n}}\mathbf{R}_S \cap \mathbf{R}$ is homogeneous, hence each $r_{\mathbf{i}}\mathbf{t}^{\mathbf{i}-\mathbf{n}} \in \mathbf{u}^{\mathbf{n}}\mathbf{R}_S \cap \mathbf{R}$. It now readily follows that each $r_{\mathbf{i}}\mathbf{t}^{\mathbf{i}} \in \mathbf{T}$ (so $\mathbf{T}$ is a $\mathbf{Z}_g$-graded subring of $R[u_1, ..., u_g, t_1, ..., t_g]$) and that $r_i \in \mathbf{u}^{\mathbf{i}}\mathbf{T} \cap R$ (so it follows that $\mathbf{T} \supseteq R[u_1, ..., u_g, \{\mathbf{t}^{\mathbf{n}}\mathbf{I}_{\mathbf{n}}\}_{\mathbf{n} \in \mathbf{N}_g}]$). Finally, if $\mathbf{P} = \{p \in \mathrm{Ass}(\mathbf{R}/\mathbf{u}^1\mathbf{R})$; there exists a depth one prime divisor of zero in the completion of $\mathbf{R}_p\}$, then it is shown in [11, (3.3)] that $\mathbf{T}$ is a finite module over $\mathbf{R}$. Q.E.D.

We can now state and prove the main result in this paper.

(3.3). THEOREM. *Let $\mathbf{I} = (I_1, ..., I_g)$ be a finite collection of ideals of a Noetherian ring R, let Γ be a multiplicatively closed set of nonzero ideals of R, and consider the following statements*:

(a) *There exists a positive integer c such that $(\mathbf{L}^1)^{c+n} : \Gamma \subseteq (\mathbf{L}^1)^n$ for all $n \geqslant 1$ and for every projective extension $\mathbf{L} = (L_1, ..., L_f)$ of $\mathbf{I}$.*

(b) *There exist an ideal H between $\mathbf{I}^1$ and $(\mathbf{I}^1)_a$ and a positive integer e such that $H^{e+n} : \Gamma \subseteq H^n$ for all $n \geqslant 1$.*

(c) *There exists an ideal K that is a projective extension of $\mathbf{I}$ such that $K^n : \Gamma = K^n$ for all $n \geqslant 1$.*

(d) *No ideal in Γ is contained in $\bigcup \{P; P \in \mathbf{E}(\mathbf{I})\}$.*

(e) *For each finite collection* $\mathbf{K} = (K_1, \ldots, K_f)$ *of ideals of* R *such that* $\mathbf{E}(\mathbf{K}) = \mathbf{E}(\mathbf{I})$ (*by* (2.2.3) *and* (2.6) *this holds if* $\mathbf{K}$ *is projectively related to* $\mathbf{I}$ *and if either* $g = f = 1$ *or each* I_i *is regular*) *it holds that* $\mathbf{S} = R[u_1, \ldots, u_f, \{\mathbf{t}^{\mathbf{n}}(\mathbf{K}^{\mathbf{n}} : \Gamma)\}_{\mathbf{n} \in \mathbf{N}_f}]$ *is a finite module over* $\mathbf{R}(R, \mathbf{K})$.

(f) *There exists a finite collection* $\mathbf{K} = (K_1, \ldots, K_f)$ *of ideals of* R *such that* $\mathbf{E}(\mathbf{K}) = \mathbf{E}(\mathbf{I})$ *and such that* $\mathbf{S} = R[u_1, \ldots, u_f, \{\mathbf{t}^{\mathbf{n}}(\mathbf{K}^{\mathbf{n}} : \Gamma)\}_{\mathbf{n} \in \mathbf{N}_f}]$ *is a finite module over* $\mathbf{R}(R, \mathbf{K})$.

(g) *For each finite collection* $\mathbf{K} = (K_1, \ldots, K_f)$ *of ideals of* R *such that* $\mathbf{E}(\mathbf{K}) = \mathbf{E}(\mathbf{I})$ *there exists* $\mathbf{h} \in \mathbf{P}_f$ *such that* $\mathbf{K}^{\mathbf{k}+\mathbf{n}} : \Gamma = \mathbf{K}^{\mathbf{n}}(\mathbf{K}^{\mathbf{k}} : \Gamma)$ *for all* $\mathbf{k} \geqslant \mathbf{h}$ *and for all* $\mathbf{n} \in \mathbf{N}_f$.

(h) *There exists a finite collection* $\mathbf{K} = (K_1, \ldots, K_f)$ *of ideals of* R *such that* $\mathbf{E}(\mathbf{K}) = \mathbf{E}(\mathbf{I})$ *and such that there exists* $\mathbf{h} \in \mathbf{P}_f$ *such that* $\mathbf{K}^{\mathbf{k}+\mathbf{n}} : \Gamma = \mathbf{K}^{\mathbf{n}}(\mathbf{K}^{\mathbf{k}} : \Gamma)$ *for all* $\mathbf{k} \geqslant \mathbf{h}$ *and for all* $\mathbf{n} \in \mathbf{N}_f$.

(i) *For each finite collection* $\mathbf{K} = (K_1, \ldots, K_f)$ *of ideals of* R *such that* $\mathbf{E}(\mathbf{K}) = \mathbf{E}(\mathbf{I})$, *there exists an ideal* $G \in \Gamma$ *such that* $\mathbf{K}^{\mathbf{n}} : \Gamma = \mathbf{K}^{\mathbf{n}} : G$ *for all* $\mathbf{n} \in \mathbf{N}_f$.

(j) *There exists a finite collection* $\mathbf{K} = (K_1, \ldots, K_f)$ *of ideals of* R *such that* $\mathbf{E}(\mathbf{K}) = \mathbf{E}(\mathbf{I})$ *and such that there exists an ideal* $G \in \Gamma$ *such that* $\mathbf{K}^{\mathbf{n}} : \Gamma = \mathbf{K}^{\mathbf{n}} : G$ *for all* $\mathbf{n} \in \mathbf{N}_f$.

Then the following hold:

(3.3.1). *If either* $g = 1$ *or each* I_i *is regular, then* (b)–(h) *are equivalent.*

(3.3.2). (e) $\Rightarrow$ (i) $\Rightarrow$ (j), *and if every ideal in* Γ *is regular, then* (j) $\Rightarrow$ (f).

(3.3.3). (a) $\Rightarrow$ (b), *and if height* $(I_i) \geqslant 1$ *for* $i = 1, \ldots, g$ *and there exists* $\mathbf{c} \in \mathbf{P}_g$ *such that* $(\mathbf{I}^{\mathbf{c}+\mathbf{n}})_a \subseteq \mathbf{I}^{\mathbf{n}}$ *for all* $\mathbf{n} \in \mathbf{N}_g$, *then* (d) $\Rightarrow$ (a).

Proof. For (3.3.1) assume that (b) holds and let $\mathbf{R} = R[u, tH]$ and $\mathbf{S} = R[u, tH_1, t^2H_2, \ldots]$, where $H_n = H^n : \Gamma$ for all $n \geqslant 1$. Then $\mathbf{R} \subseteq \mathbf{S}$, and (b) implies that $u^{e+n}\mathbf{S} \subseteq u^n\mathbf{R}$ for all $n \geqslant 1$, so $\mathbf{S}$ is a finite module over $\mathbf{R}$. Also, (3.2.1) shows that $\mathbf{S}$ is a graded subring of $R[u, t]$ whose homogeneous elements of degree n are the elements xt^n with $x \in H_n$. Therefore, if h is greater than or equal to the maximum of the degrees of a (finite) set of homogeneous linear generators of $\mathbf{S}$, considered as an $\mathbf{R}$-module, then it is readily seen that $H_{h+n} = H^nH_h$ for all $n \geqslant 1$. Let $K = H_h$, so $K = H^h : \Gamma = H^h : G$ for some $G \in \Gamma$. Then $K^n : \Gamma = (H^h : G)^n : \Gamma \subseteq (H^{nh} : G^n) : \Gamma \subseteq (H^{nh} : \Gamma) : \Gamma = H^{nh} : \Gamma = H_{nh} = H^{nh-h}H_h = (H^h)^{n-1} H_h \subseteq (H_h)^n = K^n \subseteq K^n : \Gamma$, so $K^n : \Gamma = K^n$ for all $n \geqslant 1$. Also, since $\mathbf{R} \subseteq \mathbf{S} \subseteq \mathbf{R}'$, the integral closure of $\mathbf{R}$, it is clear that $u^h\mathbf{R} \cap R \subseteq u^h\mathbf{S} \cap R \subseteq u^h\mathbf{R}' \cap R$, so $H^h \subseteq H_h$ $(= K) \subseteq (H^h)_a$. Therefore, since $\mathbf{I}^1 \subseteq H \subseteq (\mathbf{I}^1)_a$ implies that $\mathbf{I}^{\mathbf{h}} = (\mathbf{I}^1)^h \subseteq H^h \subseteq (\mathbf{I}^{\mathbf{h}})_a$, where $\mathbf{h} = h1 = (h, \ldots, h) \in \mathbf{P}_g$, it follows that $\mathbf{I}^{\mathbf{h}} \subseteq H^h \subseteq K \subseteq (H^h)_a = (\mathbf{I}^{\mathbf{h}})_a$, so K is a projective extension of $\mathbf{I}$, hence (b) $\Rightarrow$ (c).

Assume either that $g=1$ or that each I_i is regular. Also assume that (c) holds and let $G \in \Gamma$. Then it is clear that $G \nsubseteq \bigcup \{P; P \in \mathrm{Ass}(R/K^n)$ for some $n \geqslant 1\}$, and since $u^n\mathbf{R}(R, K) \cap R = K^n$ for all $n \geqslant 1$, $\bigcup \{P; P \in \mathrm{Ass}(R/K^n)$ for some $n \geqslant 1\} \supseteq \bigcup \{P; P \in \mathbf{E}(K)\}$ (by (2.1.3)) $= \bigcup \{P; P \in \mathbf{E}(\mathbf{I})\}$, the equality by either (2.2.3) or (2.6), so (c) $\Rightarrow$ (d).

Assume that (d) holds and let $\mathbf{K} = (K_1, ..., K_f)$ be a finite collection of ideals of R such that $\mathbf{E}(\mathbf{K}) = \mathbf{E}(\mathbf{I})$. Then (d) implies that: (*) no ideal in Γ is contained in $\bigcup \{P; P \in \mathbf{E}(\mathbf{K})\}$. Let $\mathbf{R} = \mathbf{R}(R, \mathbf{K})$, let S be the set of regular elements in $\mathbf{R} - \bigcup \{p; p \in \mathrm{Ass}(\mathbf{R}/\mathbf{u}^1\mathbf{R})$ and there exists a depth one prime divisor of zero in the completion of $\mathbf{R}_p\}$, and let $\mathbf{T} = \mathbf{R}_S \cap \mathbf{R}[1/\mathbf{u}^1]$. Now it follows from (*) and the definitions of S and $\mathbf{E}(\mathbf{K})$ that $\mathbf{u}^{\mathbf{n}}\mathbf{R}_S : G\mathbf{R}_S = \mathbf{u}^{\mathbf{n}}\mathbf{R}_S$ for all $\mathbf{n} \in \mathbf{N}_f$ and for all $G \in \Gamma$, so contracting to R it follows that $\mathbf{K}^{\mathbf{n}} : \Gamma \subseteq \mathbf{u}^{\mathbf{n}}\mathbf{R}_S \cap R$. It readily follows from this that if we let $\mathbf{K}_{\mathbf{n}} = \mathbf{K}^{\mathbf{n}} : \Gamma$ and $\mathbf{S} = R[u_1, ..., u_f, \{\mathbf{t}^{\mathbf{n}}\mathbf{K}_n\}_{\mathbf{n} \in \mathbf{N}_f}]$, then $\mathbf{R} \subseteq \mathbf{S} \subseteq \mathbf{T}$. Also, $\mathbf{T}$ is a finite module over $\mathbf{R}$ by (3.2.2), so $\mathbf{S}$ is a finite module over $\mathbf{R}$, so (d) $\Rightarrow$ (e).

It is clear that (e) $\Rightarrow$ (f).

The proofs that (e) $\Rightarrow$ (g) and (f) $\Rightarrow$ (h) are similar, so only the proof that (e) $\Rightarrow$ (g) will be given. For this, assume that (e) holds, let $\mathbf{K}$ be as in (e) and (g), and let $\mathbf{S}$ be as in (e), so $\mathbf{S}$ is a finite module over $\mathbf{R} = \mathbf{R}(R, \mathbf{K})$. Also, (3.2.1) shows that $\mathbf{S}$ is a $\mathbf{Z}_f$-graded subring of $R[u_1, ..., u_f, t_1, ..., t_f]$ whose homogeneous elements of degree $\mathbf{n}$ are the elements $x\mathbf{t}^{\mathbf{n}}$ with $x \in \mathbf{K}_{\mathbf{n}}$. Therefore, if $\mathbf{h} \in \mathbf{P}_f$ is greater than or equal to all the degrees of a (finite) set of homogeneous linear generators of $\mathbf{S}$, considered as an $\mathbf{R}$-module, then it is readily seen that $\mathbf{K}_{\mathbf{k}+\mathbf{n}} = \mathbf{K}^{\mathbf{n}}\mathbf{K}_{\mathbf{k}}$ for all $\mathbf{k} \geqslant \mathbf{h}$ and for all $\mathbf{n} \in \mathbf{N}_f$, so (e) $\Rightarrow$ (g) (and (f) $\Rightarrow$ (h)).

Assume either that $g=1$ or that each I_i is regular. Then either (2.2.3) or (2.6) shows that every ideal H between $\mathbf{I}^1$ and $(\mathbf{I}^1)_a$ satisfies the condition $\mathbf{E}(H) = \mathbf{E}(\mathbf{I})$, so it readily follows that (g) $\Rightarrow$ (b). Therefore if either $g=1$ or each I_i is regular, then (b), (c), (d), (e), and (g) are equivalent and (e) $\Rightarrow$ (f) $\Rightarrow$ (h), so to complete the proof of (3.3.1) it will be shown that (h) $\Rightarrow$ (c).

For this, assume that (h) holds and let $K = \mathbf{K}^{\mathbf{h}} : \Gamma$, so $\mathbf{K}^{\mathbf{h}+\mathbf{n}} : \Gamma = \mathbf{K}^{\mathbf{n}}(\mathbf{K}^{\mathbf{h}} : \Gamma) = \mathbf{K}^{\mathbf{n}}K$ for all $\mathbf{n} \in \mathbf{N}_f$. Therefore $K^n : \Gamma = (\mathbf{K}^{\mathbf{h}} : \Gamma)^n : \Gamma \subseteq \mathbf{K}^{n\mathbf{h}} : \Gamma = (\mathbf{K}^{\mathbf{h}})^{n-1}K \subseteq K^n \subseteq K^n : \Gamma$ for all $n \geqslant 1$, so (h) $\Rightarrow$ (c), hence (3.3.1) holds.

For (3.3.2) assume that (e) holds and let $\mathbf{R} = \mathbf{R}(R, \mathbf{K})$. Then there exists $\mathbf{h} \in \mathbf{P}_f$ such that $\mathbf{S} = \mathbf{A}$, where $\mathbf{A} = \sum_{0 \leqslant \mathbf{i} \leqslant \mathbf{h}} \mathbf{R} \cdot \mathbf{t}^{\mathbf{i}}(\mathbf{K}^{\mathbf{i}} : \Gamma)$. Now it is readily seen that for any finite collection $\mathbf{C}$ of ideals there exists $G \in \Gamma$ such that $C : \Gamma = C : G$ for all $C \in \mathbf{C}$. Therefore let $G \in \Gamma$ such that $\mathbf{K}^{\mathbf{i}} : \Gamma = \mathbf{K}^{\mathbf{i}} : G$ for all $\mathbf{i} \in \mathbf{N}_f$ such that $\mathbf{i} \leqslant \mathbf{h}$, so $\mathbf{A} = \sum_{0 \leqslant \mathbf{i} \leqslant \mathbf{h}} \mathbf{R} \cdot \mathbf{t}^{\mathbf{i}}(\mathbf{K}^{\mathbf{i}} : G)$. Let $\mathbf{B}$ be the Z_g-graded $\mathbf{R}$-submodule of $\mathbf{S}$ defined by $\mathbf{B} = \sum_{\mathbf{n} \in \mathbf{N}_f} \mathbf{R} \cdot \mathbf{t}^{\mathbf{n}}(\mathbf{K}^{\mathbf{n}} : G)$ (this is an $\mathbf{R}$-submodule of $\mathbf{S}$, since $\mathbf{K}^{\mathbf{n}} \subseteq \mathbf{K}^{\mathbf{n}} : G \subseteq \mathbf{K}^{\mathbf{n}} : \Gamma$ for all $\mathbf{n} \in \mathbf{N}_f$ and

$\mathbf{K}^{\mathbf{m}}(\mathbf{K}^{\mathbf{n}} : G) \subseteq \mathbf{K}^{\mathbf{m}+\mathbf{n}} : G$ for all $\mathbf{m}, \mathbf{n} \in \mathbf{N}_f$). Then is is clear that $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{S} = \mathbf{A}$, so by comparing the homogeneous components of $\mathbf{B}$ and $\mathbf{S}$ (and using (3.2.1)) it follows that $\mathbf{K}^{\mathbf{n}} : \Gamma = \mathbf{K}^{\mathbf{n}} : G$ for all $\mathbf{n} \in \mathbf{N}_f$, so (e) $\Rightarrow$ (i).

It is clear that (i) $\Rightarrow$ (j), so to complete the proof of (3.3.2) assume that (j) holds and that each $G \in \Gamma$ is regular. Let $\mathbf{S}$ and $\mathbf{R} = \mathbf{R}(R, \mathbf{K})$ be as in (f). Then it follows that $G\mathbf{S} \subseteq \mathbf{R}$, so $\mathbf{S}$ is a finite $\mathbf{R}$-module, since G is regular, hence (j) $\Rightarrow$ (f).

For (3.3.3) it is clear that (a) $\Rightarrow$ (b).

Finally assume that (d) holds, that $height(I_i) \geqslant 1$ for $i = 1, ..., g$, and that there exists $\mathbf{c} \in \mathbf{P}_g$ such that $(\mathbf{I}^{\mathbf{c}+\mathbf{n}})_a \subseteq \mathbf{I}^{\mathbf{n}}$ for all $\mathbf{n} \in \mathbf{N}_g$. Let $c = \max\{\mathbf{c}(i); i = 1, ..., g\}$. Then it is readily checked that $(\mathbf{I}^{c1+\mathbf{n}})_a \subseteq \mathbf{I}^{\mathbf{n}}$ for all $\mathbf{n} \in \mathbf{N}_g$. Let $\mathbf{L} = (L_1, ..., L_f)$ be a projective extension of $\mathbf{I}$, say $\mathbf{I}^{\mathbf{m}} \subseteq \mathbf{L}^1 \subseteq (\mathbf{I}^{\mathbf{m}})_a$ (where $\mathbf{m} \in \mathbf{P}_g$). Then by (d) and (2.3.5) it follows that, for all $G \in \Gamma$, $G \nsubseteq \bigcup \{P; P \in \hat{A}^*(\mathbf{I})\}$, so (2.5) shows that $(\mathbf{I}^{\mathbf{n}})_a : \Gamma = (\mathbf{I}^{\mathbf{n}})_a$ for all $\mathbf{n} \in \mathbf{N}_g$. Therefore $(\mathbf{L}^1)^{c+n} : \Gamma \subseteq ((\mathbf{I}^{\mathbf{m}})_a)^{c+n} : \Gamma \subseteq ((\mathbf{I}^{\mathbf{m}})^{c+n})_a : \Gamma = ((\mathbf{I}^{\mathbf{m}})^{c+n})_a \subseteq \mathbf{I}^{(c+n)\mathbf{m}-c1}$ and $c\mathbf{m} \geqslant c1$, so $\mathbf{I}^{(c+n)\mathbf{m}-c1} \subseteq (\mathbf{I}^{\mathbf{m}})^n \subseteq (\mathbf{L}^1)^n$. Thus $(\mathbf{L}^1)^{c+n} : \Gamma \subseteq (\mathbf{L}^1)^n$ for all $n \geqslant 1$, so (d) $\Rightarrow$ (a). Q.E.D.

Corollaries (3.4)–(3.7) are four useful corollaries of (3.3).

(3.4). COROLLARY. (3.4.1). *If R is a Noetherian domain, then* (3.3)(b)–(j) *are equivalent.*

(3.4.2). *If R is an analytically unramified semi-local domain, then* (3.3)(a)–(j) *are equivalent.*

Proof. (3.4.1) is clear from (3.3.1) and (3.3.2).

For (3.4.2) let $\mathbf{R} = \mathbf{R}(R, I_1, ..., I_g)$. If R is an analytically unramified semi-local domain, then the integral closure $\mathbf{R}'$ of $\mathbf{R}$ is a finite $\mathbf{R}$-module, so $\mathbf{A} = R[u_1, ..., u_g, \{\mathbf{t}^{\mathbf{n}}(\mathbf{I}^{\mathbf{n}})_a\}_{\mathbf{n} \in \mathbf{N}_g}]$ is. Also, $\mathbf{R}[1/\mathbf{u}^1] = \mathbf{A}[1/\mathbf{u}^1]$, so $\mathbf{u}^1$ is in the conductor of $\mathbf{R}$ in $\mathbf{A}$. Therefore it follows that there exists a positive integer c such that $(\mathbf{u}^1)^c \mathbf{A} \subseteq \mathbf{R}$, so $\mathbf{u}^{c1+\mathbf{n}}\mathbf{A} \subseteq \mathbf{u}^{\mathbf{n}}\mathbf{R}$ for all $\mathbf{n} \in \mathbf{N}_g$. Let $\mathbf{c} = c1 \in \mathbf{P}_g$. Then it follows that $(\mathbf{I}^{\mathbf{c}+\mathbf{n}})_a = \mathbf{u}^{\mathbf{c}+\mathbf{n}}\mathbf{A} \cap R \subseteq \mathbf{u}^{\mathbf{n}}\mathbf{R} \cap R = \mathbf{I}^{\mathbf{n}}$ for all $\mathbf{n} \in \mathbf{N}_g$. Therefore the condition in (3.3.3) holds for all $\mathbf{I} = (I_1, ..., I_g)$ when R is an analytically unramified semi-local domain, so (3.3.1)–(3.3.3) show that (3.4.2) holds. Q.E.D.

(3.5). COROLLARY. *With the notation of* (3.3) *assume that $g > 1$ and that each I_i is regular, and let* $\mathbf{H}$ *be a subset of* $\mathbf{I}$. *If one of the statements* (3.3)(b)–(h) *holds for* $\mathbf{I}$ *and Γ, then all seven statements hold for* $\mathbf{H}$ *and Γ.*

Proof. By resubscripting the I_i, if necessary, it may be assumed that $\mathbf{H} = (I_1, ..., I_f)$ with $f \leqslant g$. Since each I_i is regular, if one of (3.3)(b)–(h) holds for $\mathbf{H}$ and Γ, then (3.3.1) shows that all seven statements hold for $\mathbf{H}$ and Γ, so it suffices to show that (d) holds for $\mathbf{H}$ and Γ.

For this, let $\mathbf{A} = \mathbf{R}(R, \mathbf{H})$, $\mathbf{R} = \mathbf{R}(R, \mathbf{I})$, and $\mathbf{T} = \mathbf{A}[u_{f+1}, ..., u_g, t_{f+1}, ..., t_g]$, so $\mathbf{T} = \mathbf{R}[1/u_{f+1} \cdots u_g]$. Let $P \in \mathbf{E}(\mathbf{H})$ and by (2.1.4) let $p \in \mathrm{Ass}(\mathbf{A}/u_1 \cdots u_f \mathbf{A})$ such that the completion of $\mathbf{A}_p$ has a depth one prime divisor of zero and $p \cap R = P$. Since $t_{f+1}, ..., t_g$ are indeterminates and $u_i = 1/t_i$ for $i = f+1, ..., g$, it follows that $q = p\mathbf{T} \cap \mathbf{R} \in \mathrm{Ass}(\mathbf{R}/u_1 \cdots u_f \mathbf{R})$ and that the completion of $\mathbf{R}_q$ has a depth one prime divisor of zero. Also, since $u_{f+1}, ..., u_t$ are regular, it follows that $q \in \mathrm{Ass}(\mathbf{R}/u_1 \cdots u_g \mathbf{R})$, and it is clear that $q \cap R = p \cap R = P$, so $P \in \mathbf{E}(\mathbf{I})$ by (2.1.4), hence $\mathbf{E}(\mathbf{H}) \subseteq \mathbf{E}(\mathbf{I})$. Since (d) holds for $\mathbf{I}$ and Γ, by hypothesis and (3.3.1), it follows that (d) holds for $\mathbf{H}$ and Γ, so (b)–(h) hold for $\mathbf{H}$ and Γ by (3.3.1). Q.E.D.

If I and J are ideals in a ring R, then it is said that I and J give linearly equivalent ideal topologies on R in case there exists a positive integer h such that $I^{h+n} \subseteq J^{h+n} \subseteq I^n$ for all $n \geqslant 1$ (for example, see [2, 5, 6, 10, 11, 14, 15]). Let us generalize this as follows: let $\mathbf{I} = (I_1, ..., I_g)$ and Γ be as in (3.3) and let $\mathbf{I} : \Gamma$ denote the set $\{\mathbf{I}^{\mathbf{n}} : \Gamma;\ \mathbf{n} \in \mathbf{N}_g\}$. Then it will be said that $\mathbf{I}$ and $\mathbf{I} : \Gamma$ give *linearly equivalent ideal topologies on* R in case there exists $\mathbf{h} \in \mathbf{P}_g$ such that $\mathbf{I}^{\mathbf{h}+\mathbf{n}} \subseteq \mathbf{I}^{\mathbf{h}+\mathbf{n}} : \Gamma \subseteq \mathbf{I}^{\mathbf{n}}$ for all $\mathbf{n} \in \mathbf{N}_g$. With this terminology we have the following corollary of (3.3).

(3.6). COROLLARY. *With the notation of* (3.3) *assume either that* $g = 1$ *or that each* I_i *is regular. Then the following are equivalent*:

(3.6.1). *For every finite collection* $\mathbf{K} = (K_1, ..., K_f)$ *of ideals of* R *such that* $\mathbf{E}(\mathbf{K}) = \mathbf{E}(\mathbf{I})$, $\mathbf{K}$ *and* $\mathbf{K} : \Gamma$ *give linearly equivalent ideal topologies on* R.

(3.6.2). *There exists a finite collection* $\mathbf{K} = (K_1, ..., K_f)$ *of ideals of* R *such that* $\mathbf{E}(\mathbf{K}) = \mathbf{E}(\mathbf{I})$ *and such that* $\mathbf{K}$ *and* $\mathbf{K} : \Gamma$ *give linearly equivalent ideal topologies on* R.

Proof. This follows immediately from (3.3)(g) $\Leftrightarrow$ (h). Q.E.D.

For the next corollary of (3.3), recall that elements $b_1, ..., b_h$ in a Noetherian ring R are called an *essential sequence* in case $b_i \notin \bigcup \{P;\ P \in \mathbf{E}((b_1, ..., b_{i-1})R)\}$ for $i = 1, ..., h$. It is clear that every R-sequence is an essential sequence.

(3.7). COROLLARY. *Let* $b_1, ..., b_h$ *be an essential sequence contained in the Jacobson radical of a Noetherian ring* R, *fix* i $(0 \leqslant i \leqslant h-1)$, *let* $I = (b_1, ..., b_i)R$, *and let* $\Gamma_i = \{G;\ G$ *is an ideal in* R *and* $b_{i+1} \in \mathrm{Rad}(G)\}$. *Then* (3.3)(b)–(j) *hold for* $\mathbf{I} = (I)$ *and* $\Gamma = \Gamma_i$.

Proof. It is shown in [3, (2.3.1)] that if $b_1, ..., b_h$ are an essential sequence in a Noetherian ring R, then b_1 is regular, and it is shown in [12, (6.2)] that if each b_i is in the Jacobson radical of R, then every permutation of $b_1, ..., b_h$ is an essential sequence in R, so the hypothesis implies that I is regular and that each $G \in \Gamma$ is regular. Also, it is clear that Γ_i is a multiplicatively closed set of nonzero ideals of R, and the definitions of Γ_i and of an essential sequence show that no $G \in \Gamma_i$ is contained in $\bigcup \{P; P \in \mathbf{E}(I)\}$. Therefore (3.3)(d) holds for $\mathbf{I} = (I)$ and $\Gamma = \Gamma_i$, so (3.3.1) and (3.3.2) show that (3.3)(b)–(j) hold for $\mathbf{I} = (I)$ and $\Gamma = \Gamma_i$. Q.E.D.

Theorem (3.8) shows that if (3.3.1)–(3.3.3) hold for $\mathbf{I}$, then the analogous statements hold for $\mathbf{I}B$ for certain rings B related to R.

(3.8). THEOREM. *With the notation of* (3.3) *assume either that* $g = 1$ *or that each* I_i *is regular. Then the following statements hold*:

(3.8.1). *Let S be a multiplicatively closed set in R such that $0 \notin S$ and $I_i \cap S = \emptyset$ for $i = 1, ..., g$. If one of the statements* (3.3)(b)–(h) *holds for* $\mathbf{I}$ *and Γ, then all seven statements hold for* $\mathbf{I}R_S = (I_1 R_S, ..., I_g R_S)$ *and* $\Gamma R_S = \{GR_S; G \in \Gamma\}$. *Conversely, if $P \cap S = \emptyset$ for all $P \in \mathbf{E}(\mathbf{I})$ and if one of the statements* (3.3)(b)–(h) *holds for* $\mathbf{I}R_S$ *and* ΓR_S, *then all seven statements hold for* $\mathbf{I}$ *and* Γ.

(3.8.2). *Let Z be an ideal in R such that* $\mathrm{Ass}(R/Z) \subseteq Ass(R)$. *If one of the statements* (3.3)(b)–(h) *holds for* $\mathbf{I}$ *and* Γ, *then* (3.3)(b)–(j) *hold for* $\mathbf{I}/Z = ((I_1 + Z)/Z, ..., (I_g + Z)/Z)$ *and* $\Gamma/Z = \{(G + Z)/Z;\ G \in \Gamma\}$. *Conversely, if* $\mathrm{Ass}(R/Z) = \mathrm{Ass}(R)$ *and if one of the statements* (3.3)(b)–(j) *holds for* $\mathbf{I}/Z$ *and* Γ/Z, *then* (3.3)(b)–(h) *hold for* $\mathbf{I}$ *and* Γ.

(3.8.3). *Let A be a Noetherian faithfully flat R-algebra. Then the statements* (3.3)(b)–(h) *hold for* $\mathbf{I}$ *and* Γ *if and only if they hold for* $\mathbf{I}A = (I_1 A, ..., I_g A)$ *and* $\Gamma A = \{GA; G \in \Gamma\}$.

(3.8.4). *Let A be a finite R-algebra. If one of the statements* (3.3)(b)–(h) *holds for* $\mathbf{I}A = (I_1 A, ..., I_g A)$ *and* $\Gamma A = \{GA; G \in \Gamma\}$, *then all seven statements hold for* $\mathbf{I}$ *and* Γ. *Conversely, if* $z \cap R \in \mathrm{Ass}(R)$ *for each* $z \in \mathrm{Ass}(A)$ *and if one of the statements* (3.3)(b)–(h) *holds for* $\mathbf{I}$ *and* Γ, *then all seven statements hold for* $\mathbf{I}A$ *and* ΓA.

Proof. For (3.8.1), if one of the statements (3.3)(b)–(h) holds for $\mathbf{I}$ and Γ, then (3.3.1) shows that (3.3)(d) holds for $\mathbf{I}$ and Γ, so it follows from (2.3.1) and (3.3.1) that (3.3)(b)–(h) hold for $\mathbf{I}R_S$ and ΓR_S. And if $P \cap S = \emptyset$ for all $P \in \mathbf{E}(\mathbf{I})$ and one of (3.3)(b)–(h) holds for $\mathbf{I}R_S$ and ΓR_S, then (3.3.1) shows that (3.3)(d) holds for $\mathbf{I}R_S$ and ΓR_S, so it follows from (2.3.1) and (3.3.1) that (3.3)(b)–(h) hold for $\mathbf{I}$ and Γ.

The proof of (3.8.2) (resp., (3.8.3), (3.8.4)) is similar, but use (2.3.2) (resp., (2.3.3), (2.3.4)) in place of (2.3.1), and for (3.8.2) also use (3.3.2). Q.E.D.

This section will be closed with two corollaries of (3.8). The first combines the various parts of (3.9) for the case where R is a local ring.

(3.9). COROLLARY. *With the notation of* (3.3) *assume that R is a local ring, let R^* be the completion of R, let $z \in \mathrm{Ass}(R^*)$, and let A be a finite integral extension domain of R^*/z. Assume that one of the statements* (3.3)(b)–(h) *holds for* $\mathbf{I}$ *and* Γ. *Then* (3.3)(a)–(j) *hold for* $\mathbf{I}A$ *and* ΓA.

Proof. This follows immediately from (3.8.2)–(3.8.4) and (3.4.2), since A is a complete local domain. Q.E.D.

It is well known that R-sequences remain prime sequences in Noetherian faithfully flat extension rings. Our final corollary shows that they also behave fairly well for factor rings modulo prime divisors of zero and for finite integral extension rings.

(3.10). COROLLARY. *Let $b_1, ..., b_g$ be an R-sequence in a Noetherian ring R and for $i = 1, ..., g$ let $\Gamma_i = \{G;\ G$ is an ideal in R and $b_i \in \mathrm{Rad}(G)\}$. Then*:

(3.10.1). *Let $z \in \mathrm{Ass}(R)$ and let $'$ denote residue class modulo z. Then for $i = 0, 1, ..., g-1$ there exists an ideal C_i that is a projective extension of $(b'_1, ..., b'_i)R'$ such that $C_i^n : \Gamma'_{i+1} = C_i^n$ for all $n \geqslant 1$.*

(3.10.2). *Let B be a finite integral extension ring of R such that $z \cap R \in \mathrm{Ass}(R)$ for all $z \in \mathrm{Ass}(B)$. Then for $i = 0, 1, ..., g-1$ there exists an ideal C_i that is a projective extension of $(b_1, ..., b_i)B$ such that $C_i^n : \Gamma_{i+1}B = C_i^n$ for all $n \geqslant 1$.*

Proof. (3.10.1) (resp., (3.10.2)) follows immediately from (3.8.2) (resp., (3.8.4)) (in both cases using part (c) of (3.3)), together with the definitions of an R-sequence and the sets Γ_i. Q.E.D.

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